Supplementary Notes to Tutorial 9[1]

1 Recall

1.1 Compact Set in Euclidean Space

Let A be a subset of $\mathbb{R}^n$. Then A is *compact* if and only if A is closed and bounded subset.

1.2 Compact Surface

A *compact surface* is a surface which is also a *compact set*. A compact surface has a triangulation with a finite number of triangles.

1.3 Closed Surface

A *closed* surface is a surface that is **compact** and **without boundary**.

2 Problem related to Gauss-Bonnet Theorem

Let S be a closed surface not homeomorphic to the sphere. Show that K attains both positive and negative values.

Guideline. We will proceed the following steps to prove the statement:

- For any closed surfaces S, there is some point $p \in S$ such that $K(p) > 0$.
- Applying Gauss-Bonnet Theorem for closed surface, and then applying the classification theorem to compare the Euler characteristics $\chi(S)$.
- Since integral of K over the surface is negative, so there must be some points $q \in S$ such that $K(q) < 0$, completes the proof.

Solution.

First, since S is compact, so in particular bounded in $\mathbb{R}^3$. There exist some $R > 0$ such that $S \subseteq B_R(0)$, that is, the surface S is contained in the ball centered at the origin with radius $R > 0$. Then, we may keep shrinking the ball $B_R(0)$ until there exist at least one point of tangency between S and $B_R(0)$, say $p_0 \in S \cap B_R(0)$.

At the point p_0, since S is covered by $B_R(0)$, so we have

$$K_S(p_0) \geq K_{B_R(0)}(p_0) = \frac{1}{R^2} > 0.$$

[1] If you have any problems or spot any typos, please contact me via **maxshung.math@gmail.com**

Since S is a closed surface, hence must be compact. From the above, there is some point $p \in S$ such that $K(p) > 0$. On the other hand, since S is not homeomorphic to the sphere, by the Classification Theorem of all compact connected surfaces, we have $\chi(S) \leq 0$.

Applying the Gauss-Bonnet Theorem, we have

$$\chi(S) = \frac{1}{2\pi} \iint_S K\, dA \leq 0$$

Define two sets $S_+ := \{x \in S : K(x) > 0\}$ and $S_- := \{x \in S : K(x) \leq 0\}$. Since $S = S_+ \sqcup S_-$, and we have

$$\iint_S K\, dA = \underbrace{\iint_{S_+} K\, dA}_{\text{positive}} + \underbrace{\iint_{S_-} K\, dA}_{\text{non-positive}} \leq 0$$

Therefore, we can deduce that $|S_-| > |S_+|$, that means there are some points $q, r \in S$ such that $K(q) < 0$ and $K(r) = 0$.
Thus, K attains both positive and negative values.

3 Problem related to No Compact Minimal Surfaces

Is it possible that the mean curvature H of a compact surface S is entirely zero? Explain your answer.

Guideline. We will proceed the following steps to answer the question:

- From the section 2, we know compact surface, there is some point $p \in S$ such that $K(p) > 0$.
- Recall the principal curvatures at p is given by the eigenvalues of the shape operator, denote by κ_1, κ_2.
- From Tutorial 9, we have $H(p) = \frac{1}{2}(\kappa_1 + \kappa_2)$ and $K(p) = \kappa_1\kappa_2$.
- Conclude that no such surface exists by explaining why $H(p) \neq 0$ at $p \in S$.

Solution.

For a compact surface S, there exist some point $p \in S$ such that $K(p) > 0$. At the same point p, denote the mean curvature, two distinct principal curvatures by $H(p)$, κ_1 and κ_2 respectively.

Recall that we have

$$\begin{cases} K(p) = \kappa_1\kappa_2 \\ H(p) = \dfrac{1}{2}(\kappa_1 + \kappa_2) \end{cases}$$

From $K(p) = \kappa_1\kappa_2 > 0$, we may deduce that κ_1, κ_2 are of the same sign, either both positive or both negative. So, it follows that

$$H(p) = \frac{1}{2}(\kappa_1 + \kappa_2) \neq 0$$

which means H of S is not entirely zero.

Thus, it is impossible to find such compact minimal surface in $\mathbb{R}^3$.